A finite element analysis of spatial solitons in optical fibres

A. Nicolet, F. Drouart and G. Renversez
Faculté de Saint-Jérôme, Institut Fresnel, Université Paul Cézanne Aix-Marseille III, Marseille Cedex, France, and
C. Geuzaine
Department of Electrical Engineering and Computer Science, Institut Montefiore, Université de Liège, Liège, Belgique

Abstract

Purpose – This paper concerns the study of non-linear effects in optical fibres with a core made of a Kerr type medium. The aim is to propose an algorithm to find spatial solitons, i.e. solutions with an harmonic behaviour in time and along the fibre but with a field distribution in the cross section corresponding to a self-trapped propagation of the electromagnetic field.

Design/methodology/approach – The field is supposed to be harmonic in time and along the direction of invariance of the fibre but inhomogeneous in the cross section. This modifies the refractive index profile of the fibre (a step-index one in this study). A scalar model of the fibre together with the finite element method (that is well suited to deal with inhomogeneous media) are used and a new iterative algorithm is proposed to obtain the non-linear solutions. An adaptive meshing is necessary to guarantee the accuracy of the model.

Findings – The new algorithm converges to self-coherent solutions that are different from the ones obtained via a fixed power algorithm. Both the equivalent of a fundamental mode and of a second order mode are studied.

Originality/value – The approach allows the findings of the previously known spatial solitons (with a slight modification of our algorithm) together with a new family of solutions. It opens a new field of investigation to understand this whole family of non-linear solutions as it shows that only a small part of them where known up to now.

Keywords Non-linear control systems, Finite element analysis method, Optical cables, Approximation theory

Paper type Research paper

Introduction

Non-linear optical effects may be used to improve characteristics of optical fibres or to create new behaviour. In an optical Kerr medium, the relative electric permittivity is of the form $\varepsilon_r = n_0^2 + n_2^2\langle E^2\rangle$ where $\langle E^2\rangle$ is a quadratic time average of the field, for instance the square of the field amplitude in the harmonic case (Kivshar and Agrawal, 2003). The refractive index is higher where the electric field has the largest value and it is expected that the non-linear medium acts as a convex lens. Propagating solutions with a better confinement than in the linear case may therefore be expected.

In the linear case, the propagating modes in a waveguide (i.e. a structure invariant along the z-axis and therefore completely described by its cross section in the xy-plane) are the solutions of the form $\mathscr{E} = \Re\mathrm{e}\{\mathbf{E}(x,y)\mathrm{e}^{-i(\omega t-\beta z)}\}$ that give a complete description of the possible solutions as any signal propagating along the fibre is a linear combination of such modes. Contrarily to the linear case, a complete description of the propagating signals in non-linear media is not possible and only particular solutions are accessible. Given two

COMPEL: The International Journal for Computation and Mathematics in Electrical and Electronic Engineering
Vol. 26 No. 4, 2007
pp. 1105-1113

0332-1649
DOI 10.1108/03321640710756429

particular solutions of a non-linear problem, their sum is by no way itself a solution. A usual approach used to study the propagation in non-linear optical fibres is to neglect the distribution of the electromagnetic field in the cross section and to consider only the variation along the fibre with respect to time *t*. For instance, the non-linear Schrödinger (NLS) equation is then obtained and particular solutions are the so-called temporal solitons (Kivshar and Agrawal, 2003).

On the contrary, in this paper, we are interested in the distribution of the electromagnetic field in the cross section of the fibre and in particular solutions called the spatial solitons (Ferrando *et al.*, 2003). We describe in details the numerical method to obtain these non-linear solutions. Then, we illustrate it via a simple case of the step index fibre with a non-linear Kerr type core.

Spatial solitons

We consider the scalar model for the propagating solution obtained under the weak guidance (weak refractive index contrast) hypothesis (Zolla *et al.*, 2005). In this case, the electric field $\mathcal{E}$ is supposed to have only a non-vanishing transverse component of known (arbitrary) direction given by the unit vector $\hat{\mathbf{e}}$. Moreover, its divergence is usually neglected so that $\mathrm{div}(\mathcal{E}) = 0$ is assumed.

In the linear case, the electric field corresponding to a propagation mode is therefore a field of the form:

$$\mathcal{E} = \Re\mathrm{e}\{\phi(x, y)\mathrm{e}^{-i(\omega t - \beta z)}\}\hat{\mathbf{e}} \tag{1}$$

in which $\omega = k_0 c$ is the pulsation and β is the propagation constant. The problem reduces to the determination of the function $\phi(x, y)$ and of the constant β for a given value of k_0 by solving the scalar equation:

$$\Delta_t \phi + k_0^2 \varepsilon_r \phi = \beta^2 \phi \tag{2}$$

obtained from the Maxwell equations with materials of relative permittivity ε_r and permittivity μ_0 and using all the hypotheses here above. Δ_t is the transverse Laplacian. The dispersion curves are the set of pairs (k_0, β) for which a solution of equation (2) exists.

In the non-linear case where the fibre is made of optical Kerr material, the relative permittivity is itself a function of the field intensity and the following dependence is assumed: $\varepsilon_r(\phi) = n_0^2 + n_2^2|\phi|^2$ where n_0 (the linear refractive index) and n_2 (the Kerr coefficient) are constants characterizing the material (Boyd, 2003).

As the nonlinearities depend only on the modulus of the field and not on its instantaneous value, it may be possible to obtain solutions that can be represented by equation (1) and this will be our fundamental hypothesis. Such solutions, if they exist, are the spatial solitons (Ferrando *et al.*, 2003). We are therefore looking for solutions (β, ϕ) of the non-linear equation:

$$\Delta_t \phi + k_0^2 \left(n_0^2 + n_2^2|\phi|^2\right)\phi = \beta^2 \phi \tag{3}$$

Numerical algorithm

The numerical method used to solve equation (3) is the finite element method (Zolla *et al.*, 2005) that is well adapted since it easily takes into account inhomogeneous permittivities.

In the present case of a scalar model, the very classical approximation based on the piecewise linear interpolation on a triangular mesh is used.

In the case where a single Kerr material is used, setting the reduced field $\phi_r = n_2\phi$ allows to reduce equation (3) to:

$$\Delta_t \phi_r + k_0^2\left(n_0^2 + |\phi_r|^2\right)\phi_r = \beta^2\phi_r \quad (4)$$

which is independent of the Kerr coefficient. Clearly, this means that the refractive index profile leading to the self-coherent solution ϕ_r depends on the linear part of the medium but not on the value of the Kerr coefficient n_2: only the quadratic dependence does matter. The physical field $\phi = \phi_r/n_2$, however, depends on the coefficient n_2 and this one the lower, the larger has to be the power injected to produce the self-coherent solution.

The solutions are supposed to be close to the modes of the linear fibre and therefore the proposed algorithm is a simple Picard iteration in which a propagation mode is computed in a linear fibre with a refractive index profile determined by the field intensity obtained at the previous iteration.

The starting point is the linear fibre with an homogeneous index. For a given k_0, some modes are computed (by solving a matrix eigenvalue problem to find the β's and the corresponding electric fields) and the mode of interest is selected, e.g. the fundamental mode. The corresponding electric field (whose amplitude is arbitrarily fixed in the linear fibre only) is used to compute the new refractive index profile, then new modes are computed with this given refractive index. The mode of interest is selected and used to modify again the refractive index profile that gives a new eigenvalue problem. This process is repeated until the refractive index profile reaches a fixed point!

This process seems quite simple but there is a fundamental flaw: the amplitude of eigenmodes is irrelevant and the numerical solutions of the intermediary eigenvalue problems used in the process here above have an uncontrolled amplitude. On the contrary, the non-linear problem depends fundamentally on the amplitude of the field and therefore, this amplitude has to be determined a posteriori for the mode of interest. The chosen solution $\psi(x, y)$ of the numerical eigenvalue problem has thus to be scaled by a scalar factor χ to obtain the reduced field $\phi_r = \chi\psi$. A suitable numerical value of χ may be obtained via the annulment of a weighted residual of equation (4) with the solution ψ itself taken as the weight factor (to minimize the local error where the field has the highest values).

First, (ψ_i, β_i) at step i are computed as particular solutions to the eigenvalue problem:

$$\Delta_t \psi_i + k_0^2\left(n_0^2 + |\phi_{r,i-1}|^2\right)\psi_i - \beta_i^2\psi_i = 0 \quad (5)$$

Then, the value of χ_i is computed in order to optimize the self-coherence of $\phi_{r,i} = \chi_i\psi_i$ by annulling the residue:

$$\int_K \left(\Delta_t \psi_i + k_0^2\left(n_0^2 + |\chi_i\psi_i|^2\right)\psi_i - \beta_i^2\psi_i\right)\bar{\psi}_i \,\mathrm{d}S = 0 \quad (6)$$

where the integral is computed on the cross section K of the Kerr medium region. Using directly this equation leads to an ill-conditioned expression for χ_i because two terms of very similar magnitude are subtracted. With the identity:

$$\left(\Delta_t \psi_i + k_0^2 n_0^2 \psi_i - \beta_i^2 \psi_i\right) \bar{\psi}_i = k_0^2 |\phi_{r,i-1}|^2 |\psi_i|^2 \tag{7}$$

deduced from equation (5), a numerically well-conditioned expression for χ_i is obtained:

$$\chi_i^2 = \frac{\int_K |\psi_i|^2 |\phi_{r,i-1}|^2 \mathrm{d}S}{\int_K |\psi_i|^4 \mathrm{d}S} \tag{8}$$

The whole process is summarized in the following algorithm:

- **begin**:
- Set $\psi_0 = 0$, $\chi_0 = 1$, $i = 1$
- **repeat**

  -Compute the eigenfunctions ψ_i (and the corresponding β_i's) via the finite element solution of the eigenvalue problem defined by equation (5) and select the one of interest (e.g. the fundamental one).

  -Compute χ_i via formula (8).

  -Set $i \leftarrow i + 1$
- **until** the variation between (ψ_i, β_i) and $(\psi_{i-1}, \beta_{i-1})$ is small enough.
- The $(\chi_{\mathrm{coh.}} \psi_{\mathrm{coh.}} / n_2 = \phi_{\mathrm{coh.}}, \beta_{\mathrm{coh.}})$ of the last iteration is the self coherent solution.
- **end**.

As shown by this approach, the proposed algorithm renormalizes the field at each step of iteration and therefore at the convergence we can deduce the power a posteriori (defined as the integral of $\chi^2_{\mathrm{coh.}} \psi^2_{\mathrm{coh.}}$. However, a "fixed-power" algorithm can be used in which the power is given a priori (Ferrando *et al.*, 2003; Fujisawa and Koshiba, 2003). To study this well-known algorithm, a simple modification must be done in the formula (8):

$$\chi_i^2 = \frac{P}{\int_K |\psi_i|^2 \mathrm{d}S}$$

where P is the fixed value of the power. By the way, this proves the flexibility of the implementation of our approach.

Numerical example

We consider the simple case of a cylindrical fibre with a Kerr material core of radius 2 μm and with the linear part of the refractive index n_0 (Silica) = 1.45 embedded in an "infinite" matrix with a linear refractive index $n = 1.435$ (weak contrast). We set also the Dirichlet condition at the edge of the geometry (and remark that in the present paper we are not yet interested by the computation of the leaky modes (Zolla *et al.*, 2005).

First of all, we study the convergence of our algorithm. Figure 1 shows that there is a fast convergence: for instance, the final relative precision on effective index is about 10^{-6} for ten steps. In addition, this behaviour is true for all the considered range of wavelengths (0.25 μm $< \lambda <$ 2.5 μm).

Figure 2(a) shows the field distribution for the fundamental mode in the linear case (corresponding to the first iteration of our process) and Figure 2(b) shows the field distribution with a Kerr nonlinearity.

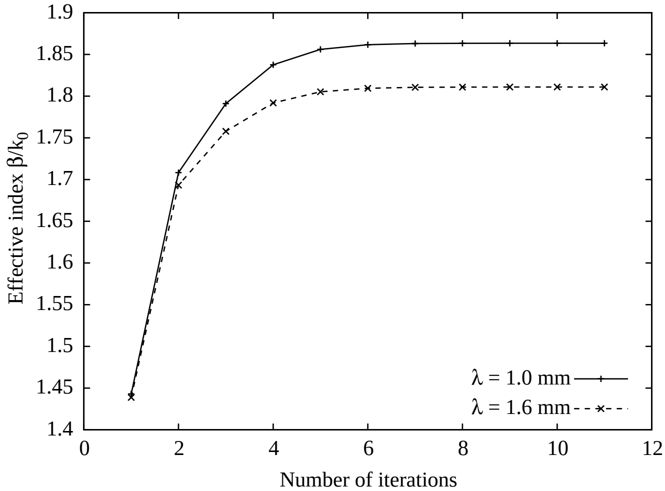

Figure 1. Algorithm convergence for the fundamental mode at $\lambda = 1.0$ and $1.6\,\mu$m

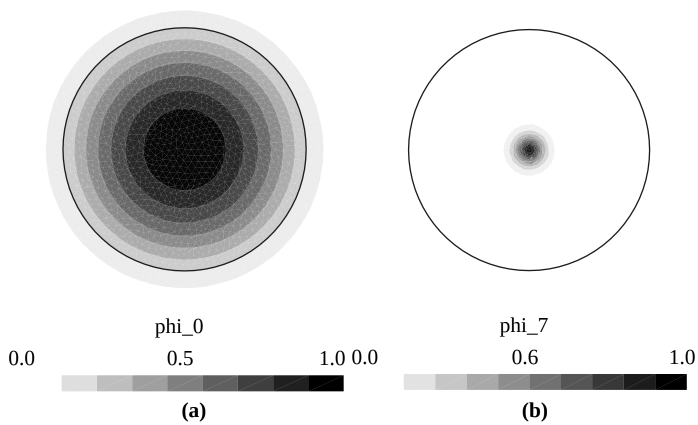

Figure 2. Field distribution in linear (a) and nonlinear (b) cases for the fundamental mode at $\lambda = 1.0\,\mu$m

Also, it is interesting to study the Kerr effect in the second mode which is less confined than the fundamental mode (Figure 3(a)).

We notice that the non-linear effect involves a strong confinement in the fibre core. Moreover, in the second mode, the shape of the field distribution is modified with the Kerr effect (Figure 3(b)).

Consequently, the field is significantly different from zero only on a small number of elements (Figure 4). That corresponds to an inefficient use of degrees of freedom and for that reason, the propagation constant is mis-evaluated (Figure 5) even if we use a fine mesh.

Indeed, Figure 5 shows the evolution of the effective index $n_{\text{eff.}} = \beta/k_0$ with various meshes: a very fine one with more than 20,000 elements and a coarse one with nearly 4,000 elements or less. We notice that for small wavelengths, the computed effective index depends on the number of elements because the field confinement is stronger there.

To solve this issue and to describe the field correctly without a huge mesh, we use an adaptive mesh refinement (Figure 6).

Figure 7 shows how we use the adaptive mesh and how we increase the algorithm speed. Experience has shown that a single refinement based on the knowledge of the solution of the linear case is sufficient to provide a satisfactory solution. Moreover, further refinement based on next steps of the process tends to lead to excessively concentrated meshing. In addition, Figure 5 shows that for $\lambda > 1.5\,\mu$m (it is assumed here that the wavelength is given for free space), mesh is less influential, therefore a fine mesh is not necessary. Therefore, with this method, as the non-linear field is more confined, we are sure that the mesh computed with the information provided by the linear field covers perfectly the non-linear field.

The size of the elements is prescribed by the function $a|1 - b|\psi||^c$ in which a (μm), b (without unit) and c (without unit) are positive adjustable parameters and ψ is the linear normalized field.

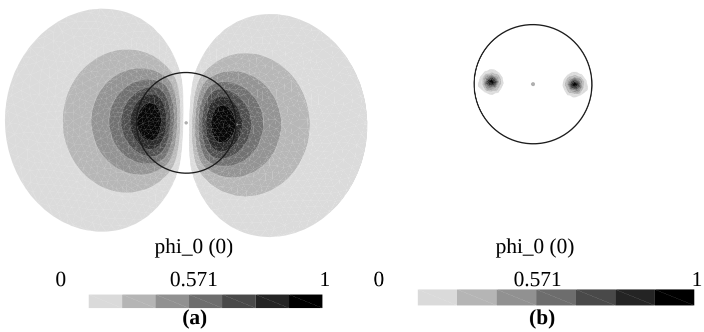

Figure 3.
Field distribution in linear (a) and nonlinear (b) cases for the second mode at $\lambda = 1.0\,\mu$m

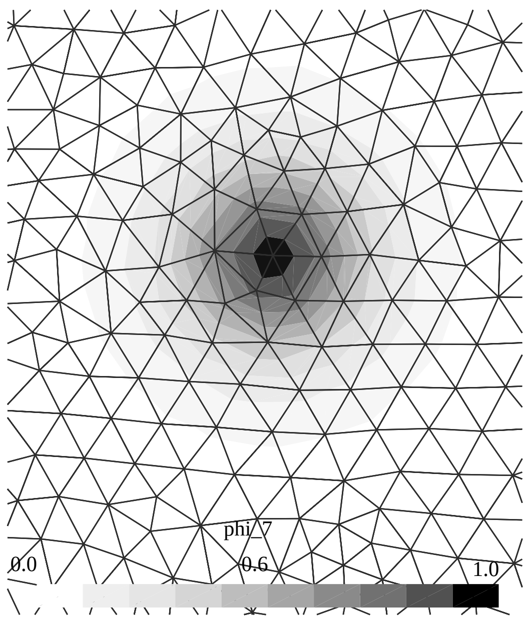

Figure 4.
Non adaptive mesh (with 12,000 elements) with the non-linear field distribution at $\lambda = 0.5\,\mu$m

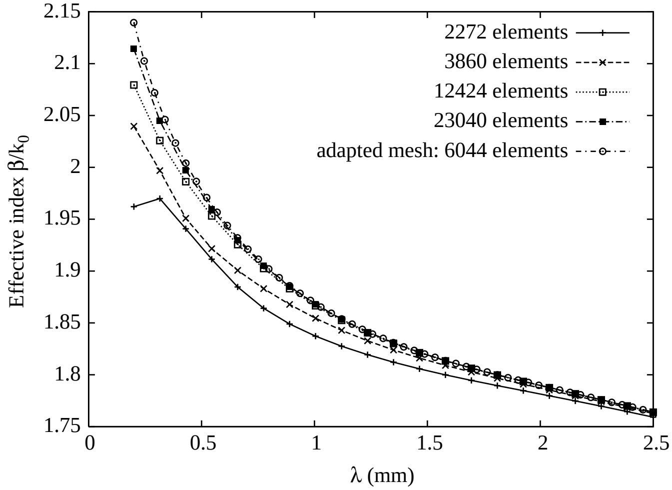

Figure 5.
Non-linear effective index with various meshes and with the adaptive mesh

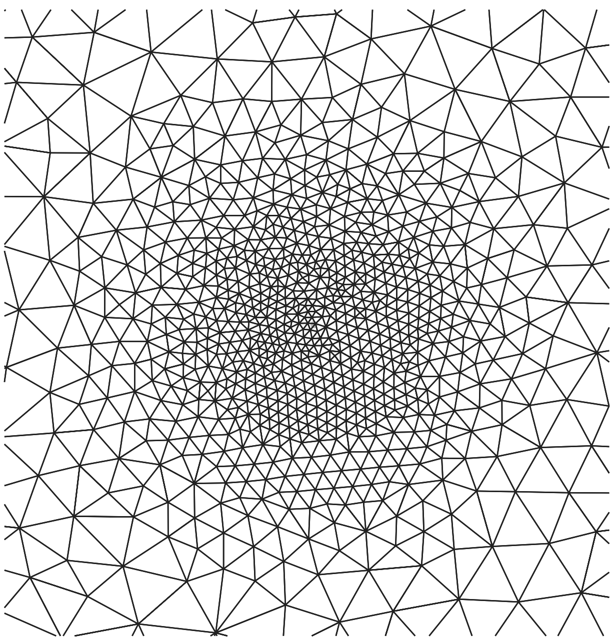

Figure 6.
Adaptive mesh (with only 6,000 elements) at $\lambda = 0.5\,\mu$m

We set these parameters according to λ. For instance, to have lots of elements for high-field and few elements for low-field at $\lambda = 0.2\,\mu$m, the values of b and c must tend toward 1 and 2, respectively, and $3 < a < 10\,\mu$m.

As shown by Figure 5, the use of the adaptive mesh allows us to compute the effective index accurately with only 6,000 elements. Therefore, the proposed algorithm

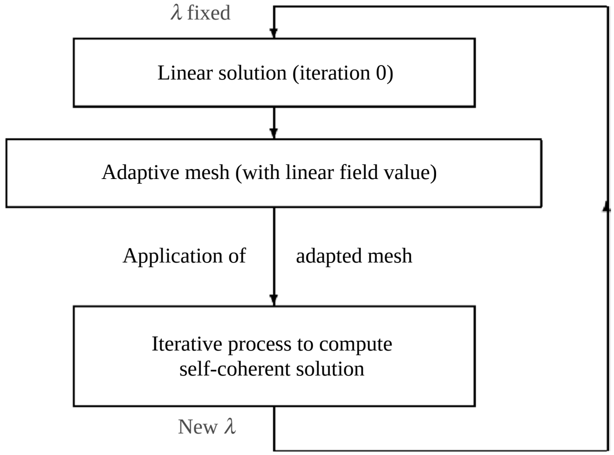

Figure 7.
Scheme to use the adaptive mesh

gives, via the adaptive mesh, both accurate and fast results for the fundamental mode and higher order modes.

Figure 8 shows the variation of the fundamental mode effective index β/k_0 versus λ. It appears that the variation of the non-linear index curve is more important than the one of the linear index curve.

Figure 9 shows the variation, for the fundamental mode, of the relative difference between the effective index $n_{\mathrm{eff.}}^{\mathrm{NL}}$ of the non-linear self-coherent solutions and the effective index of the corresponding modes of the linear fibre $n_{\mathrm{eff.}}^{\mathrm{L}}$ with respect to the wavelength. It shows that the larger the wavelength, the weaker the non-linear effect.

Conclusions and future works

The finite element method proved to be a very flexible method in order to manage the inhomogeneity of the permittivity/refractive index induced by the optical Kerr effect.

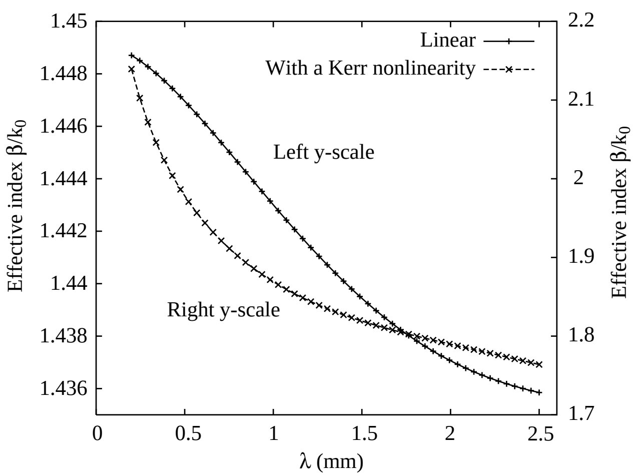

Figure 8.
Effective index in the linear case and in the non-linear one for the fundamental mode

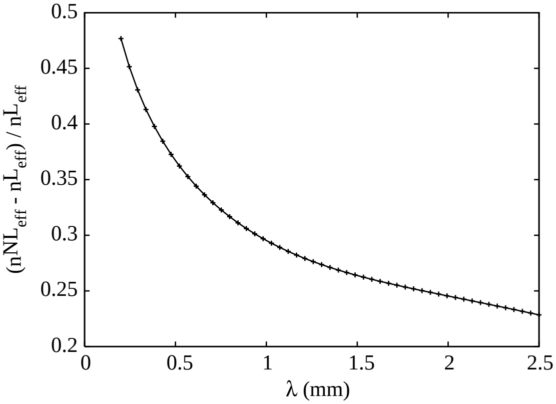

Figure 9. Non-linear effect for the fundamental mode effective index according to the wavelength

The proposed algorithm proved to be quite stable to find the self-coherent solution (spatial soliton) corresponding to the fundamental mode in a step-index fibre. Via the adaptive mesh, the results are obtained quickly and accurately. Finally, we illustrate our method on a simple step index fibre with a Kerr type core.

However, our algorithm seems sensitive to the shape and the amplitude of the initial field ψ_0 used to start the algorithm. This issue allows us to believe that there is a continuum of solutions as it is the case of the low-power solutions found by the fixed power algorithm (Ferrando *et al.*, 2003; Fujisawa and Koshiba, 2003).

Future works will concern:

- a systematic study by varying the shape and the amplitude of the initial solution ψ_0.
- the development of the full vector model to get rid from the simplification hypotheses necessary in the scalar approximation.
- the study of microstuctured optical fibres (MOFs). Our final aim is to study with this algorithm the influence of the Kerr nonlinearity in the MOFs.

References

Boyd, R.W. (2003), *Nonlinear Optics*, 2nd ed., Academic Press, Amsterdam.

Ferrando, A., Zacarès, M., De Córdoba, P.F., Binosi, D. and Monsoriu, J.A. (2003), “Spatial soliton formation in photonic crystal fibres”, *Optics Express*, Vol. 11 No. 5, pp. 452-9.

Fujisawa, T. and Koshiba, M. (2003), “Finite element characterization of chromatic dispersion in nonlinear holey fibers”, *Optics Express*, Vol. 11 No. 13, pp. 1481-9.

Kivshar, Y.S. and Agrawal, G.P. (2003), *Optical Solitons*, Academic Press, Amsterdam.

Zolla, F., Renversez, G., Nicolet, A., Kuhlmey, B., Guenneau, S. and Felbacq, D. (2005), *Foundations of Photonic Crystal Fibres*, Imperial College Press, London.

Corresponding author

A. Nicolet can be contacted at: andre.nicolet@fresnel.fr